

# Impact of Single Links in Competitive Percolation

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## Abstract

How a complex network is connected crucially impacts its dynamics and function [1–4]. Percolation, the transition to extensive connectedness upon gradual addition of links, was long believed to be continuous [5] but recent numerical evidence on „explosive percolation“ [6] suggests that it might as well be discontinuous if links compete for addition. Here we analyze the microscopic mechanisms underlying discontinuous percolation processes and reveal a strong impact of single link additions. We show that in generic competitive percolation processes, including those displaying explosive percolation, single links do not induce a discontinuous gap in the largest cluster size in the thermodynamic limit. Nevertheless, our results highlight that for large finite systems single links may still induce observable gaps because gap sizes scale weakly algebraically with system size. Several essentially macroscopic clusters coexist immediately before the transition, thus announcing discontinuous percolation. These results explain how single links may drastically change macroscopic connectivity in networks where links add competitively.

Percolation, the transition to large-scale connectedness of networks upon gradual addition of links, occurs during growth and evolutionary processes in a large variety of natural, technological, and social systems [1]. Percolation arises in atomic and molecular solids in physics as well as in social, biological and artificial networks [2, 7–10]. In the more complex of these systems, adding links often is a competitive process. For instance, a human host carrying a virus may travel at any given time to one but not to another geographic location and therefore infect other people only at one of the places [11, 12]. Across all percolating systems, once the number of added links exceeds a certain critical value, extensively large connected components (clusters) emerge that dominate the system.

Given the breadth of experimental, numerical, and empirical studies, as well as several theoretical results and analytic arguments [13–16], percolation was commonly believed to exhibit a continuous transition where the relative size of the largest cluster increases continuously from zero in the thermodynamic limit once the number of links crosses a certain threshold. So recent work by Achlioptas, D’Souza and Spencer [6] came as a surprise because it suggested a new class of random percolating systems that exhibit “explosive percolation” [17]. Close to some threshold value, the system they considered displays a steep increase of the largest cluster size with increasing the number of links; moreover, numerical scaling analysis of finite size systems suggests a discontinuous percolation transition. This study initiated several follow-up works (e.g. [10, 18–25]) confirming the original results for a number of system modifications. These in particular support that competition in the addition of links is crucial; the key mechanisms underlying discontinuous percolation, however, are still not well understood and the impact of individual link additions is unknown.

Gaining one or a few links may have drastic consequences for a network’s growth and its overall dynamics, depending on whether or not such individual links qualitatively alter the global connectivity of a network. For instance, spontaneous activity in developing neural circuits may become persistent after establishing some additional synaptic connections [26, 27]. Similarly, during beginning pandemics the specific travel patterns of a single infected person may substantially change the number of infecteds on a time scale of months [11].

Here we identify how microscopic single-link additions impact competitive processes. We find that in generic percolation processes, single links do not induce macroscopic gaps in the largest cluster size as the system size  $N \rightarrow \infty$ . Nevertheless, the gap sizes decay weakly algebraically as  $N^{-\beta}$  with often small  $\beta$  such that gaps are essentially macroscopic, i.e.

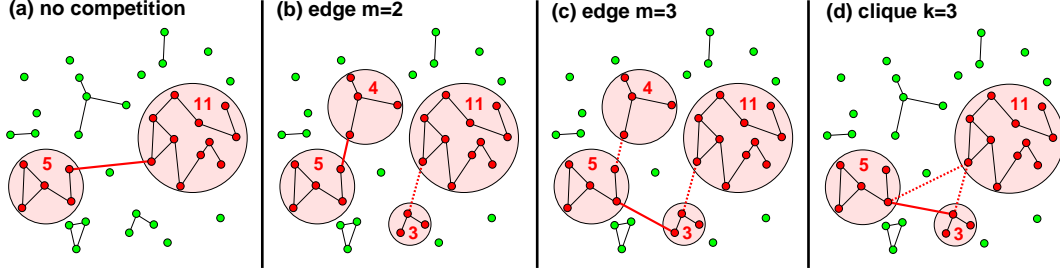


Figure 1: Competitive vs. non-competitive percolation processes. (a) Non-competitive Erdős-Rényi percolation: new randomly chosen links just add. (b) Edge competition:  $m = 2$  links compete with each other and clusters of sizes 4 and 5 win the competition and join to form a new cluster of size 9 (c)  $m = 3$  links compete with each other. Clusters of sizes 3 and 5 join. (d) Clique competition ( $k = 3$ ): three links within a clique compete. Clusters of sizes 3 and 5 join. Throughout all panels, small disks indicate nodes, solid black lines existing links; large shaded disks indicate clusters entering the competition with numbers denoting their sizes; red-dashed lines indicate potentially new, competing links; solid red lines indicate actual link added.

substantially large even for systems of macroscopic size  $N \approx 10^{23}$ . Such gaps, induced by single links, occur at the point of percolation transitions, are a key signature of discontinuous percolation, and are announced by several coexisting, essentially macroscopic clusters.

### The nature of discontinuities in competitive percolation processes

Consider a family of competitive percolation processes where potentially new links compete with others for addition (Fig. 1). Starting with an empty graph of a large number  $N$  of isolated nodes (no links), links sequentially add in competition with others. For *edge competition*, for each single-link addition,  $m$  potential links are randomly selected. The link for which the sum of the cluster sizes containing their two end-nodes is smallest wins the competition and adds. Intra-cluster links are possible; these can only broaden the transition compared to disallowing them. For  $m = 1$ , this process is non-competitive and identical to random Erdős-Rényi percolation [14], whereas for  $m = 2$  it specializes to the process introduced before [6]. For all  $m \geq 2$ , this kind of competition promotes that during gradual addition of links smaller clusters tend to be connected (to form larger ones) before larger clusters grow. With increasing  $m$ , the competition becomes more strongly competitive, be-

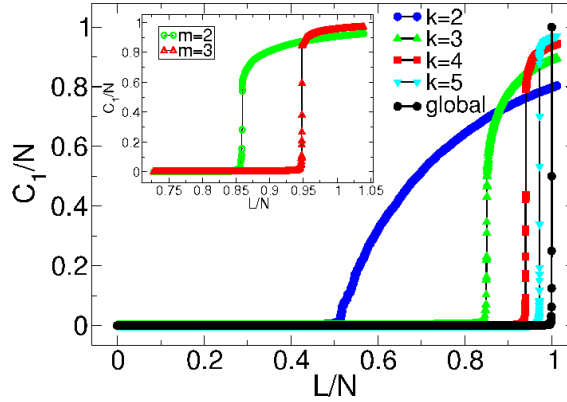


Figure 2: Growth of the largest cluster size  $C_1$  as a function of the number of added links  $L$  for non-competitive ( $k = 2$ ), competitive, and globally competitive percolation processes for both edge (inset) and clique competition (main panel);  $N = 2^{16}$  nodes, quantities on both axes rescaled by system size  $N$ . Single realization is displayed for each percolation process.

cause more potentially new links actually compete. If  $m$  is maximal, all potential links in the network compete for addition and we have *global competition*.

Taking the sum of cluster sizes in edge competitive processes appears somewhat arbitrary because, e.g., taking the product [28], or, for that matter, any convex function of the two cluster sizes, has similar competitive effects, cf. [6]. We thus consider also *clique competition* that does not suffer from this ambiguity. For clique competition, randomly draw a fixed number  $k$  of nodes and connect those two of them contained in the two smallest clusters. Here  $k = 2$  describes non-competitive random percolation and for all  $k \geq 3$  competition has the same principal effect on changes in cluster sizes as edge competition. We remark that for maximal possible  $k$  we again have global competition.

For large finite systems, single realizations of genuinely competitive processes ( $m \geq 2$  or  $k \geq 3$ , cf. Fig. 2) exhibit macroscopic  $\mathcal{O}(N)$  changes in the size  $C_1$  of the largest cluster  $\mathbf{C}_1$ . In fact, numerical scaling studies (Supplementary Fig. 1) confirm that the transition regime in the plane spanned by  $\ell = L/N$  and  $c_1 = C_1/N$  shows an  $\mathcal{O}(1)$  change of  $c_1$  in a region of width  $\Delta\ell$  that scales as  $N^{-\gamma}$ ,  $\gamma > 0$ , for large  $N$  (cf. also [6]). These results may suggest that in the limit of infinite systems there is a discontinuous  $\mathcal{O}(1)$  gap in the curve characterizing competitive percolation in the  $\ell - c_1$  plane.

Further investigating the microscopic dynamics of the transition, however, seeds doubt about any such gap. If the largest gap  $\Delta C_{\max} := \max_L (C_1(L+1) - C_1(L))$  is macroscopic (extensive),

$$\lim_{N \rightarrow \infty} \frac{\Delta C_{\max}}{N} > 0, \quad (1)$$

we call such transitions *strongly discontinuous*, otherwise *weakly discontinuous* (see Supplementary Information for an exact definition). For weakly discontinuous transitions, the curve in the  $\ell - c_1$  plane does in fact not exhibit any macroscopic gap in the thermodynamic limit.

Evaluating the largest jump size  $\Delta C_{\max}$  from extensive numerical simulations of systems up to size  $2^{26} \approx 6.7 \times 10^7$  already suggests (Fig. 3) that it scales algebraically as

$$\frac{\Delta C_{\max}}{N} \sim N^{-\beta} \quad (2)$$

independent of whether the process is non-competitive, minimally competitive ( $k = 3$ ,  $m = 2$ ) or exhibits even stronger forms of competition ( $k \geq 4$ ,  $m \geq 3$ ). As we find that  $\beta > 0$  for all such processes, we have  $\lim_{N \rightarrow \infty} \Delta C_{\max}/N = 0$  and thus the transitions are all weakly discontinuous. The only exception seems to be global competition where we find  $\beta$  indistinguishable from zero and  $\Delta C_{\max}/N \approx 0.5 > 0$  for all  $N$  (Fig. 1), suggesting a strongly discontinuous transition. The set of all numerical analyses therefore suggests that competitive percolation transitions are generically weakly discontinuous, and single links do not induce a gap in  $c_1$  in the thermodynamic limit  $N \rightarrow \infty$ . Nevertheless, as the gap sizes scale weakly algebraically with system size (2) with often small  $\beta$  such gaps may still be essentially macroscopic, i.e. substantially large even for macroscopic systems of large finite size  $N$ .

### The impact of single links

So how can single links actually impact the dynamics of the transition? For the extreme case of global competition, exact analytical arguments reveal the occurrence of macroscopic jumps and gives key insights about the nature of transitions in competitive percolation processes, that similarly hold for weakly discontinuous transitions (see below): We label all existing clusters by  $\mathbf{C}_i$  and their sizes by  $C_i = |\mathbf{C}_i|$  where the index  $i$  enumerates their size rank such that  $C_1 \geq C_2 \geq \dots \geq C_{\nu_{\max}}$  where  $\nu_{\max} \leq N$  denotes the total number of existing

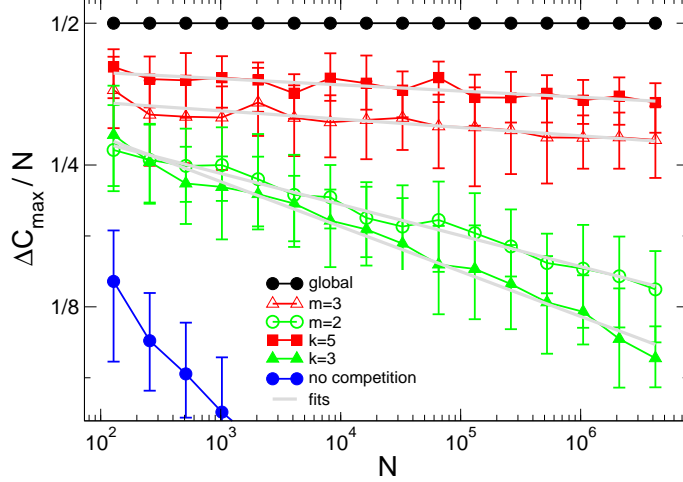


Figure 3: Gap sizes  $\Delta C_{\max}$  decay algebraically with system size  $N$  for weakly discontinuous transitions. Green symbols: weakest competition; red symbols: stronger competition. black symbols: global competition; blue symbols: no competition ( $k = 2$ ,  $m = 1$ ). The symbols indicate average values for 50 realizations; error bars indicate 25%-quantiles and reflect system-intrinsic fluctuations. Solid grey lines are best least-square fits (slopes,  $\beta = 0.013$  ( $k = 5$ )  $\beta = 0.021$  ( $m = 3$ ),  $\beta = 0.065$  ( $m = 2$ ) and  $\beta = 0.095$  ( $k = 3$ )). Black line shows the analytical curve for global competition, where  $\beta = 0$ .

clusters. For global competition each newly added link joins the two smallest clusters in the entire system such that  $C_{\nu_{\max}} + C_{\nu_{\max}-1} \rightarrow C'$ . For simplicity of presentation, we choose the system size  $N$  to be a power of 2. This ensures that up to  $L_1 = N/2$  new links only connect 1-clusters (isolated nodes) to result in new 2-clusters (two nodes with a single connecting link) such that the maximum cluster size stays  $C_1 = 2$  for all  $L \leq L_1$ . The subsequent  $N/4$  links each connect 2-clusters to 4-clusters, keeping  $C_1 = 4$  until  $L_2 = 3N/4$ . In general, new links added between  $L_{n-1}$  and  $L_n$  connect  $n$ -clusters to  $2n$ -clusters keeping  $C_1 = 2^n$  where  $L_n = \frac{(2^n-1)}{2^n}N$ , for all  $n \leq \log_2(N)$ . In the final step, at  $L = N - 1$ , the remaining two  $\frac{N}{2}$ -clusters join and induce the largest gap

$$\frac{\Delta C_{\max}}{N} = \frac{1}{2}, \quad (3)$$

analytically confirming the numerical findings (Figs. 2 and 3). As a consequence, global competition (involving information about the entire system's state for local link addition) implies a genuine gap of size  $1/2$  in the main order parameter  $c_1$ .

For weaker forms of competition, closely related link adding mechanisms control the cluster joining dynamics. Inspecting the impact of single link additions onto cluster joining dynamics in more detail we identify three distinct mechanisms that may contribute towards increasing the size  $C_1$  of the current largest cluster in more general competitive processes:

- (i) *Largest cluster growth*: the largest cluster itself connects to a smaller cluster of size  $C_i < C_1$  and grows,  $\mathbf{C}_1 + \mathbf{C}_i \rightarrow \mathbf{C}_1$ , to stay the largest cluster.
- (ii) *overtaking*: two smaller clusters of size  $C_i, C_j < C_1$  join into one that is larger than the current largest cluster,  $\mathbf{C}_i + \mathbf{C}_j \rightarrow \mathbf{C}_1$ , and the originally largest cluster becomes second largest,  $\mathbf{C}_1 \rightarrow \mathbf{C}_2$ ;
- (iii) *doubling*: if there are several clusters of maximal size  $C_1 = C_2 = \dots = C_\nu$  for some  $\nu \geq 2$ , two of these join,  $\mathbf{C}_i + \mathbf{C}_j \rightarrow \mathbf{C}_1$  for some  $i, j \in \{1, \dots, \nu\}$ , creating a new largest cluster of twice the size of the original one.

For each single link addition, we denote the probability for normal cluster growth (i) by  $p_{\text{gr}}$ . We say that  $p_{\text{gr}} = 0$  if the probability of normal cluster growth (i) is zero up to the point where only two clusters are left in the system and normal growth is the only remaining way the largest cluster could grow at all (see Supplementary Information for a more formal definition).

As we show in the following, an arbitrary percolation process with  $p_{\text{gr}} = 0$  necessarily exhibits a genuine gap and thus a strongly discontinuous transition, i.e.  $\Delta C_{\text{max}}/N$  stays positive in the limit of infinitely large system sizes  $N$ . As growth (i) is prohibited, the largest cluster size changes either by overtaking (ii) or by doubling (iii). During any such percolation process adding a link never more than doubles  $C_1$ . As a consequence, there is a certain  $L'$  such that  $C_1(L')$  is larger than  $N/3$  but not larger than  $2N/3$ . When  $\mathbf{C}_1$  will be overtaken (or doubles) one more time at some  $L = L' + \Delta L$ , the cluster previously largest becomes the second largest,  $\mathbf{C}_1 \rightarrow \mathbf{C}_2$  (or disappears in case of doubling). Thus it is guaranteed that during percolation two clusters of sizes  $C_1 \geq N/3$  and  $C_2 \geq N/3$  are generated which necessarily join at some time  $L > L'$ . Therefore, in any such competitive process, prohibited growth  $p_{\text{gr}} = 0$  implies that the largest gap is macroscopic,

$$\frac{\Delta C_{\text{max}}}{N} \geq \frac{1}{3}. \quad (4)$$

Hence, all competitive percolation processes with  $p_{\text{gr}} = 0$  display strongly discontinuous transitions with a strong impact of single link additions. As we show in the Supplementary Information, such a gap necessarily occurs at or beyond  $\ell_c = 1$ ; thus for extremal competition with  $p_{\text{gr}} = 0$  the percolation point, where the largest cluster becomes macroscopic, does not necessarily coincide with the point where the largest gap occurs.

### Single links induce gaps in large finite systems

Nevertheless, many weakly discontinuous transitions still exhibit essentially macroscopic gaps for large finite systems: We conjecture that competitive percolation processes in nature (or engineering or the social world), in particular spatially extended systems with limited range interactions shall naturally allow the largest cluster to grow,  $p_{\text{gr}} > 0$  (as do all competitive percolation processes for non-global clique and edge competition) and they generically exhibit weakly discontinuous (if not continuous) percolation transitions [29, 30]. In specific limiting models analytic mean field considerations yield

$$\frac{\Delta C_{\text{max}}}{N} \sim N^{-\beta}, \quad \beta > 0, \quad (5)$$

thus confirming (2). For instance, in a model variant where largest cluster joins with the smallest available with probability  $p$ , and otherwise the two smallest clusters join with probability  $1 - p$  we analytically find that (see Supplementary Information for a detailed derivation)

$$\beta = 1 + \frac{\log(2)}{\log[(1-p)/(2-p)]} \approx \frac{p}{2 \log(2)} \quad (6)$$

for  $0 \leq p \ll 1$  scales roughly linearly with  $p$ .

Notably, if largest cluster growth does not occur,  $p = 0$ , we have  $\beta = 0$  and  $\Delta C_{\text{max}}/N > 0$  in the thermodynamic limit, consistent with Eq. (3). More importantly, these results show that even if the largest cluster may grow the slightest, i.e. for the smallest possible size increase with arbitrarily small  $p > 0$  the percolation transition is weakly discontinuous, because  $\beta > 0$  as soon as  $p \neq 0$ . Direct numerical simulations well agree with our analytical prediction (6), see Figure 4. The finding that  $\beta > 0$  as soon as  $p > 0$  is consistent with the above general result that for arbitrarily small probability  $p_{\text{gr}} > 0$  of cluster growth, the percolation transition is already weakly discontinuous, often with small positive exponents  $\beta$



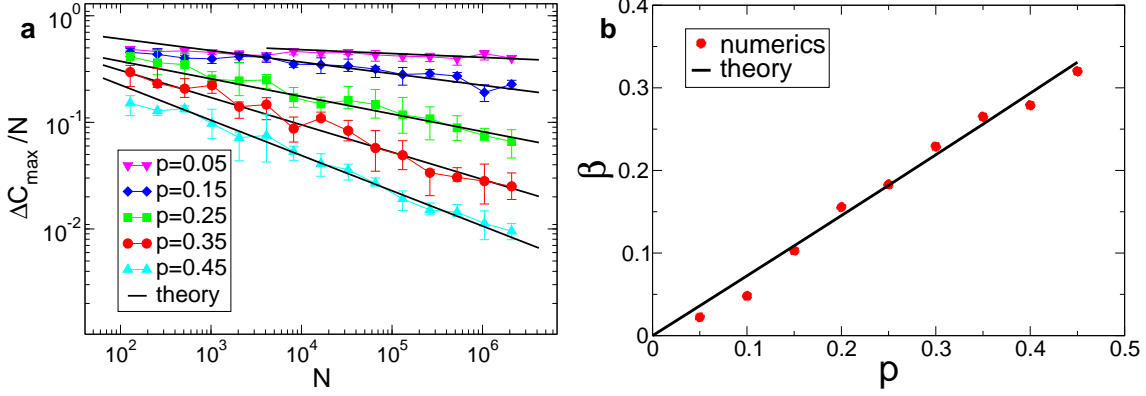


Figure 4: Weakly discontinuous transition in stochastic mixture of largest cluster growth (with probability  $p$ ) and suppressed growth. (a) Double-logarithmic plot of  $\Delta C_{\max}/N$  vs.  $N$  for different  $p$ . The slopes of the theoretical mean field prediction (black lines; ranging from  $\beta = 0.036$  ( $p = 0.05$ ) to  $\beta = 0.33$  ( $p = 0.45$ )) asymptotically well fit the gap sizes obtained by numerical experiments (symbols). (b) Indeed, the theoretically derived exponent  $\beta$  (6) as a function of  $p$  (no fit parameter) systematically well predicts those found from fitting the data in (a) (red dots).

and thus essentially macroscopic gaps in large finite systems (see numerical example below). More generally, the results above suggest that any process with non-maximal competition (including non-maximal edge competition ( $m = 2$ ) displaying “explosive percolation” [6, 18, 31, 32]) generically displays weakly discontinuous transitions.

### Finite size scaling and coexisting large clusters

Further extensive numerical scaling analysis reveals that the gaps in the generic competitive percolation processes we consider indeed occur coincident with the point where the largest cluster size is discontinuous (Supplementary Figure 2). Moreover, immediately before the transition, not only the largest gap size, but also the second largest cluster, the third largest cluster etc. appear essentially macroscopic (Fig. 5). In particular, the maximum second largest cluster generically exactly equals the maximum gap size,  $\Delta C_{\max} = \max_L C_2(L)$ ; see Supplementary Information for a derivation. Thus for small  $\beta$  the largest cluster is essentially non-unique, in contrast to standard continuous percolation transitions. Finally, analytical arguments also demonstrate that the percolation strength [19, 24], defined as the difference in largest cluster size immediately after and immediately before the gap, exactly

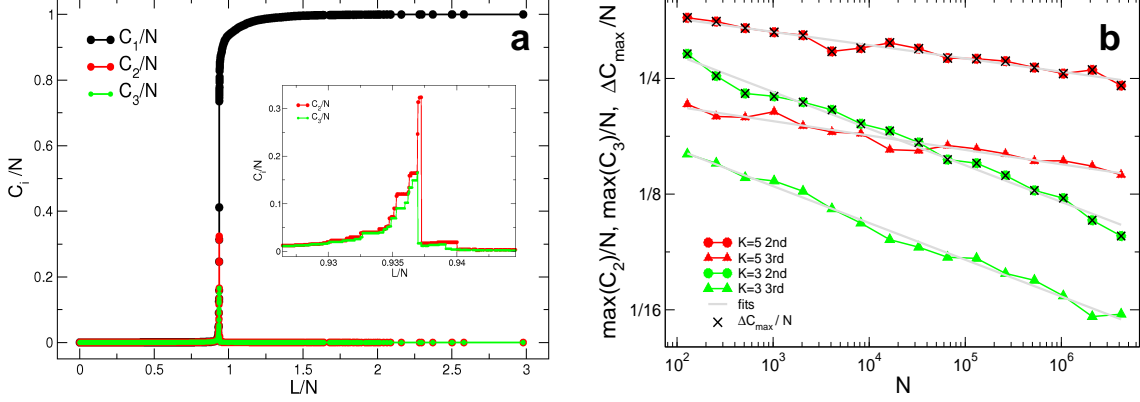


Figure 5: Several large clusters coexist at a discontinuous percolation transition. (a) Main panel: Simultaneous emergence of the largest cluster of size  $C_1$ , the second largest cluster  $C_2$ , and the third largest cluster  $C_3$  in a competitive percolation process (clique percolation,  $k = 4$ ,  $N = 2^{15}$ ). Inset: blow-up of  $C_2$  and  $C_3$  in the region around the transition point. (b) The maximal sizes of second and third largest clusters as a function of network size  $N$  indicate that they have the same order of magnitude and the same scaling that is moreover identical to that of  $\Delta C_{\max}$ . In particular,  $\max_L C_i(L) \sim \Delta C_{\max} \sim N^{-\beta}$  for  $i \in \{2, 3\}$  with  $\beta = 0.095 \pm 0.001$  for  $k = 3$  and  $\beta = 0.036 \pm 0.001$  for  $k = 5$ . The maximum gap size  $\Delta C_{\max}$  ( $\times$ ) in fact exactly equals the maximum size of the second largest cluster. Thus, there is no unique large cluster right at the transition even for very large finite systems.

equals the size of the second largest cluster before the transition, which in turn scales with the same exponent  $\beta$  as the gap size (2). Taken together, single link additions induce several new distinctive features of discontinuous percolation transitions and thus serve as a key mechanism controlling competitive percolation processes.

Interestingly, the so-called  $k$ -cores of the evolving graph, serving as the key example of the drastic impact of single links in traditional percolation theory [33, 34], exhibit dynamics very similar to that for Erdős-Rényi percolation, even for extreme processes with  $p_{gr} = 0$ . The  $k$ -core of a graph is the largest subgraph with minimum degree at least  $k$ . As numerical simulations indicate (Supplementary Figure 6), the size of the 2-core increases continuously from zero whereas  $k$ -cores for all  $k \geq 3$  exhibit a discontinuous jump induced by single link additions. These results hold for both Erdős-Rényi as well as competitive percolation processes. Even for extreme processes with  $p_{gr} = 0$  the 2-core is still continuous, but with the location of the transition moved to larger values compared to the point of percolation.

The dynamics of  $k$ -cores is thus very similar for competitive and standard, non-competitive percolation processes, in stark contrast to the dynamics of the largest cluster size. This is true even though, as shown above, the latter is also strongly influenced by single link additions.

## Discussion

These results explain how microscopic mechanisms of single-link addition control the dynamics of the size of the largest cluster and impact the type of transition. In particular, the exponent  $\beta$  tells in how far single link additions change macroscopic connectivity. For generic competitive processes  $\beta$  is smaller than for non-competitive ones (see Fig. 3), but our numerical and analytic results indicate that they are still distinct from zero. Only processes with global competition or other extreme forms of competition yield  $\beta = 0$  and thus a discontinuous gap  $\Delta C_{\max}$  induced by single link addition. Others, more generic processes, typically exhibit  $\beta > 0$  and thus a weakly discontinuous transition.

It is important to note that percolation processes with only moderate competition may already yield very small positive exponents and thus essentially macroscopic gaps (see Fig. 2). Here we used “essentially macroscopic” to mean that (a) the addition of single links in systems of physically large size induces gaps that are of relevant size (substantial fraction of system size) and that (b) the gap sizes increase with stronger competition (e.g. increasing  $k$ ) yielding a decreasing exponent  $\beta \rightarrow 0$  as  $k \rightarrow N$ . As a consequence, even processes actually exhibiting weakly discontinuous transitions may display large gaps in systems of physically relevant size (compare with Fig. 3). For instance, if  $\beta = 0.02$ , a system of macroscopic, but finite size  $N = 10^{23}$  exhibits a gap of  $\Delta C_{\max}/N \sim N^{-\beta} \approx 0.35$  although formally  $\Delta C_{\max}/N \rightarrow 0$  as  $N \rightarrow \infty$ . For many real processes with already moderate forms of competition, we expect exponents  $\beta$  close to zero, and thus conjecture that single links may have a strong impact onto how such a network becomes connected.

In summary, our results demonstrate how in competitive percolation, keeping the growth rate of the largest cluster small, strengthens the impact of single link additions that merge smaller clusters. Growing (i) and overtaking (ii) markedly distinguish the microscopic dynamics in systems exhibiting competitive percolation. The more largest cluster growth is suppressed, the more relevant the discontinuous gap becomes in large systems of given finite

size. Single link additions may then induce an essentially macroscopic gap even for weakly discontinuous transitions if competition is sufficiently strong.

Interestingly, a protein homology network has recently been identified [10] that displays macroscopic features akin to explosive percolation. Individual links may also induce abrupt changes in several other growing networked systems, possibly with severe consequences for the systems' dynamics and function (compare to [33–35]). For instance, growing one or a few additional synaptic connections in a neuronal circuit may strongly alter the global connectivity and thus the overall activity of the circuit [26, 27]; specific infected individuals traveling to one but not another location may drastically change the patterns of infectious diseases [11]; and the macroscopic properties of complex systems exhibiting competitive aggregation dynamics of physical or biological units may exhibit abrupt phase transitions induced by a small set of specific individual bonds newly established, compare, e.g. [36, 37]. Our study thus does not only provide recipes (by looking for certain competitive cluster formation) to identify real systems that could exhibit a (weakly) discontinuous percolation transition, but also shows that and how single link additions in such systems may induce discontinuous gaps, and in turn a collective, very abrupt change of structure and dynamics.

The current study answers how single-link dynamics underlies competitive percolation in general, but does not tell how single link additions are actually generated and controlled in any given real system. Future work must bridge this gap and establish how competitive percolation, and in particular the creation of essentially macroscopic jumps due to single link additions, is influenced by predefined structure, e.g. for percolation processes on lattices and in geometrical or topological confinement occurring in nature [10, 18–20, 22].

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